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C^*-norms defined by positive linear forms

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It is well known that the norm of a C^*-algebra A is uniquely defined by a set of its positive linear forms, namely

$$\|x\| = \sup_{f\in S} \sqrt{f(x^*x)} \tag{1}$$

for all $x\in A$, where S is the set of positive linear forms f on A such that $\|f\|\leqq 1$ (cf. [1]). It is worth mentioning that the set S in (1) can be defined by an entirely algebraic way; the linear form f on A is in S if and only if $f(x^*x)\in\mathbf{R}_+$ and $|f(x)|^2\leqq\leqq f(x^*x)$ for all $x\in A$. This means that in the right side of (1) only such entities enter which are related to the algebraic structure of A.

The latter observation leads to the result that defining a C^*-norm on a $*$-algebra is equivalent to indicating a certain set of positive linear forms on it. So the question arises naturally; how to choose a set S of positive linear forms on a given $*$-algebra so as to obtain a C^*-norm on it by the definition (1)?

The aim of this paper is to examine this problem. Due to the elementary character of our further investigations we shall constantly refer to the basic works [1] and [2].

I. C^*-seminorms defined by positive linear forms

In the present study the vector space of the linear forms on the $*$-algebra A is denoted by A^* and $\widetilde{\text{co}}\,(P)$ stands for the $\sigma(A^*, A)$-closed convex hull of the subset P of A^*.

If A is a $*$-algebra with unity and P is a set of positive linear forms on A then the set $\{f\in P\,|\,f(\mathbf{1})\leqq 1\}$ is denoted by $P(1)$, where $\mathbf{1}$ is the unit element of A. Further, assuming that $P(1)$ is $\sigma(A^*, A)$-bounded, $\|\cdot\|_P$ denotes the mapping from A into $\mathbf{R}_+$

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defined by the equation

$$\|x\|_P := \sup_{f \in P(1)} \sqrt{f(x^*x)} \tag{1'}$$

for all $x \in A$. It is obvious that $\|\cdot\|_P$ is a seminorm on A. The dual seminorm of $\|\cdot\|_P$ is denoted by $\|\cdot\|'_P$.

Given a $*$-algebra A and a linear form f on it, for all $x \in A$ we define the linear forms $x.f$ and $f.x$ on A as the mappings $y \mapsto f(xy)$ and $y \mapsto f(yx)$, respectively. Clearly, $x.(f.y)=(x.f).y$ thus one may use the simple notation $x.f.y$ instead of $x.(f.y)$ or $(x.f).y$.

Our first result concerns a sufficient condition on a set of positive linear forms providing C^*-seminorms by the aid of (1)′.

Theorem 1. *Let A be a $*$-algebra with unity and P a nonvoid set of positive linear forms on A satisfying*

(I) *$P(1)$ is $\sigma(A^*, A)$-bounded.*

(II) *$\mathbf{R}_+P \subset P$ and $x^*.P.x \subset \widetilde{\mathrm{co}}\,(P)$ for all $x \in A$.*

Then $\|\cdot\|_P$ is a C^*-seminorm on A and every $f \in \widetilde{\mathrm{co}}\,(P)$ is $\|\cdot\|_P$-continuous and $\|f\|'_P = f(1)$. Moreover, if P separates the points of A then $\|\cdot\|_P$ is a norm on A and the involution of A is proper.

Proof. Since $\widetilde{\mathrm{co}}\,(P)$ is nonvoid and $\sigma(A^*, A)$-bounded, we may form the seminorm $\|\cdot\|_{\widetilde{\mathrm{co}}(P)}$ besides $\|\cdot\|_P$. We claim that these seminorms are equal. Of course, $\|\cdot\|_P \leqq \|\cdot\|_{\widetilde{\mathrm{co}}(P)}$. In order to prove the reverse inequality it suffices to show that $f(x^*x) \leqq \|x\|_P^2$ for all $x \in A$ and $f \in (\widetilde{\mathrm{co}}\,(P))(1)$, $f \neq 0$. If $f \in (\widetilde{\mathrm{co}}\,(P))(1)$ then there is generalized sequence $(f_i)_{i \in I}$ in $\mathrm{co}\,(P)$ such that $f_i \to f$ pointwise on A. In particular, $f_i(1) \to f(1)$ and $f(1) > 0$, thus we may suppose that $f_i(1) > 0$ for all $i \in I$. Then take $f_i' := (f(1)/f_i(1))f_i$ $(i \in I)$. Clearly, $f_i' \in (\mathrm{co}\,(P))(1)$ with regard to the first part of (II) and it is easy to see that $\mathrm{co}\,(P(1)) = (\mathrm{co}\,(P))(1)$. On the other hand, $f_i' \to f$ pointwise on A. From this we infer that given an element x of A we have $f_i'(x^*x) \to f(x^*x)$ and $f_i'(x^*x) \leqq \|x\|_P^2$, since $f_i' \in \mathrm{co}\,(P(1))$, thus $f(x^*x) \leqq \|x\|_P^2$. Now fix an element f in $(\widetilde{\mathrm{co}}\,(P))(1)$. Then

$$|f(x)|^2 \leqq f(1)f(x^*x) \leqq \|x\|^2_{\widetilde{\mathrm{co}}(P)} = \|x\|_P^2$$

for all $x \in A$, i.e. f is $\|\cdot\|_P$-continuous and $\|f\|'_P \leqq 1$. The first part of (II) now yields that $\mathbf{R}_+\widetilde{\mathrm{co}}\,(P) \subset \widetilde{\mathrm{co}}\,(P)$, hence f is $\|\cdot\|_P$-continuous and $\|f\|'_P = f(1)$ for all $f \in \widetilde{\mathrm{co}}\,(P)$.

We show next that

$$f(y^*x^*xy) \leqq \|x\|_P^2 f(y^*y) \tag{2}$$

for all $f \in \widetilde{\mathrm{co}}\,(P)$ and $x, y \in A$. If $f(y^*y) = 0$ then choose an arbitrary positive real number ε to obtain

$$f(y^*x^*xy)/\varepsilon = ((y/\sqrt{\varepsilon})^*.f.(y/\sqrt{\varepsilon}))(x^*x).$$

Since by (II) the linear form standing on the right hand side belongs to $(\overline{\mathrm{co}}\,(P))(1)$, we now have $f(y^*x^*xy)\leqq\varepsilon\|x\|_P^2$ for all $\varepsilon>0$, i.e. the equality $f(y^*x^*xy)=0=$ $=\|x\|_P^2 f(y^*y)$ holds. If $f(y^*y)>0$, then

$$f(y^*x^*xy)/f(y^*y) = ((y/\sqrt{f(y^*y)})^*.f.(y/\sqrt{f(y^*y)}))(x^*x)$$

and the linear form on the right hand side belongs to $(\overline{\mathrm{co}}\,(P))(1)$, hence we obtain the desired inequality.

From the inequality (2) it follows that $\|xy\|_P\leqq\|x\|_P\|y\|_P$ for all $x, y\in A$. The only thing that remained to be proved is the inequality $\|x\|_P^2\leqq\|x^*x\|_P$ for all $x\in A$. If $x\in A$ and $f\in P(1)$ then $f(x^*x)\leqq\|f\|_P'\|x^*x\|_P=f(1)\|x^*x\|_P\leqq\|x^*x\|_P$ thus $\|\cdot\|_P$ is a C^*-seminorm on A. The last assertion of the theorem is an immediate consequence of our previous considerations.

II. C^*-norms defined by positive linear forms

The assumptions (I) and (II) introduced in Theorem 1 are not sufficient for P to ensure that $\|\cdot\|_P$ be a C^*-norm on A. Our next aim is to impose further conditions on P in order to have possibility of proving the completeness of A with respect to the uniform structure defined by $\|\cdot\|_P$.

We need two auxiliary lemmas. Before formulate them we agree that given a $*$-algebra A with unity and a nonvoid set P of positive linear forms on A satisfying (I), the linear subspace and the $\|\cdot\|_P'$-closed linear subspace of A^* spanned by P will be denoted by $\mathrm{sp}\,(P)$ and $\overline{\mathrm{sp}}\,(P)$, respectively.

Lemma 1. *Let A be a $*$-algebra with unity and P a separating set of positive linear forms on A satisfying* (I). *Then the $\sigma(A, \overline{\mathrm{sp}}\,(P))$ and $\sigma(A, \mathrm{sp}\,(P))$ topologies coincide in each $\|\cdot\|_P$-bounded subset of A.*

Proof. Let $(x_i)_{i\in I}$ be a generalized sequence in A such that $x_i\to x$ in the $\sigma(A, \mathrm{sp}\,(P))$-topology and suppose that we have $M:=\sup_{i\in I}\|x_i\|_P<+\infty$. We claim that $x_i\to x$ in the $\sigma(A, \overline{\mathrm{sp}}\,(P))$-topology. If $\bar f\in\overline{\mathrm{sp}}\,(P)$, $f\in\mathrm{sp}\,(P)$ and $i\in I$ then

$$|\bar f(x_i)-\bar f(x)| \leqq |\bar f(x_i)-f(x_i)|+|f(x_i)-f(x)|+|f(x)-\bar f(x)| \leqq$$
$$\leqq \|\bar f-f\|_P'(M+\|x\|_P)+|f(x_i)-f(x)|.$$

Since $\bar f\in\overline{\mathrm{sp}}\,(P)$ and $f(x_i)\to f(x)$ for all $f\in\mathrm{sp}\,(P)$, we easily deduce that $\bar f(x_i)\to$ $\to\bar f(x)$.

It is obvious that $\sigma(A, \overline{\mathrm{sp}}\,(P))$ is a stronger topology on A than $\sigma(A, \mathrm{sp}\,(P))$, however, Lemma 1 indicates a certain strict intrinsic relation between them.

Lemma 2. *Let A and P be as in Lemma 1 and suppose*

(III) $x.P \subset \overline{\mathrm{sp}}\,(P)$ *for all* $x \in A$.

Then the multiplication in the algebra A is $\|\cdot\|_P$-boundedly left and right continuous in the $\sigma(A, \mathrm{sp}\,(P))$-topology.

Proof. Consider a $\|\cdot\|_P$-bounded generalized sequence $(x_i)_{i \in I}$ in A such that $x_i \to x$ in the $\sigma(A, \mathrm{sp}\,(P))$-topology. Then combine Lemma 1 and (III) to obtain that

$$f(yx_i) = (y.f)(x_i) \to (y.f)(x) = f(yx)$$

for all $f \in P$ and $y \in A$, i.e. $yx_i \to yx$ in the $\sigma(A, \mathrm{sp}\,(P))$-topology.

Due to the obvious equality $f.y = (y^* .f)^*$ and the trivial fact that the set $\overline{\mathrm{sp}}\,(P)$ is closed with respect to the natural adjunction of A^*, we deduce that $x_i y \to xy$ in the same topology.

We are now ready to formulate our main result on C^*-norms defined by positive linear forms.

Theorem 2. *Let A be a $*$-algebra with unity and P a separating set of positive linear forms on A satisfying* (I), (II), (III) *and*

(IV) *A is sequentally complete with respect to the uniform structure defined by the $\sigma(A, \mathrm{sp}\,(P))$-topology.*

Then A is a C^*-algebra whose (unique) C^*-norm equals $\|\cdot\|_P$.

Proof. With regard to Theorem 1, A, equipped with $\|\cdot\|_P$ is a pre-C^*-algebra. Let $\hat{A}$ denote the envelopping C^*-algebra of this pre-C^*-algebra. For every $\|\cdot\|_P$-continuous linear form f on A let $\hat{f}$ denote its unique norm continuous extension to $\hat{A}$. We shall prove that $A = \hat{A}$.

First we show that there exists a unique mapping π from $\hat{A}$ onto A with the property $f \circ \pi = \hat{f}$ for all $f \in P$. The uniqueness of π is an immediate consequence of the assumption that P separates the points of A. To prove the existence of π let us consider a fixed element $\hat{x}$ of $\hat{A}$. There is a sequence $(x_n)_{n \in \mathbf{N}}$ in A such that $\|x_n - \hat{x}\|_P \to 0$, where the C^*-norm of $\hat{A}$ is also denoted by $\|\cdot\|_P$. Then $(x_n)_{n \in \mathbf{N}}$ is a $\|\cdot\|_P$-Cauchy sequence in A, hence a Cauchy sequence in the weaker uniform structure defined by the $\sigma(A, \mathrm{sp}\,(P))$-topology. Now, condition (IV) implies the existence of an element x in A with the property that $x_n \to x$ in the $\sigma(A, \mathrm{sp}\,(P))$-topology. We define $\pi(\hat{x})$ to be equal to x. Then, for all $f \in P$, by virtue of Theorem 1 we have

$$f(\pi(\hat{x})) = f(x) = \lim_n f(x_n) = \lim_n \hat{f}(x_n) = \hat{f}(\hat{x}),$$

thus the existence of π is verified.

We claim that $f \circ \pi = \hat{f}$ holds for all $f \in \overline{\mathrm{sp}}\,(P)$. Indeed, if $f \in \overline{\mathrm{sp}}\,P$ and $(f_n)_{n \in \mathbf{N}}$ is a sequence in $\mathrm{sp}\,(P)$ such that $\|f_n - f\|'_P \to 0$ then we also have $\|\hat{f}_n - \hat{f}\|'_P \to 0$, where $\| \cdot \|_P'$ denotes the dual norm of the C^*-norm of $\hat{A}$. Thus $\hat{f}_n \to \hat{f}$ pointwise on A, hence

$$f(\pi(\hat{x})) = \lim_n f_n(\pi(\hat{x})) = \lim_n \hat{f}_n(\hat{x}) = \hat{f}(\hat{x})$$

for all $\hat{x} \in \hat{A}$, thus providing the desired equality.

Obviously, π is linear and it is continuous in the topology defined by the C^*-norm on $\hat{A}$ and $\sigma(A, \overline{\mathrm{sp}}(P))$ on A, respectively. It is easy to see that π preserves the involution of $\hat{A}$ and A, respectively. On the other hand, $\pi \circ \pi = \pi$, since $f(x) = \hat{f}(x) = f(\pi(x))$ for all $f \in P$ and $x \in A$, i.e. we have $x = \pi(x)$ $(x \in A)$.

We show that the map π is multiplicative, thus π, in fact, is a $*$-algebra morphism between $\hat{A}$ and A. Let $x \in A$ and $\hat{y} \in \hat{A}$. Applying (III) we infer $x.f \in \overline{\mathrm{sp}}(P)$ for every $f \in P$, hence

$$f(\pi(x\hat{y})) = \hat{f}(x\hat{y}) = (x.\hat{f})(\hat{y}) = (\widehat{x.f})(\hat{y}) = (x.f)(\pi(\hat{y})) = f(x\pi(\hat{y}))$$

i.e. $\pi(x\hat{y}) = x\pi(\hat{y})$. Next, let $\hat{x} \in \hat{A}$ and $\hat{y} \in \hat{A}$. Then there is a sequence $(x_n)_{n \in \mathbf{N}}$ in A such that $\|x_n - \hat{x}\|_P \to 0$. The above established continuity property of π now yields $x_n = \pi(x_n) \to \pi(\hat{x})$ in the $\sigma(A, \mathrm{sp}(P))$-topology (moreover, in $\sigma(A, \overline{\mathrm{sp}}(P))$). Since $\pi(x_n\hat{y}) = x_n\pi(\hat{y})$ $(n \in \mathbf{N})$ and the sequence $(x_n)_{n \in \mathbf{N}}$ is $\| \cdot \|_P$-bounded in A, Lemma 2 implies that $\pi(x_n\hat{y}) \to \pi(\hat{x})\pi(\hat{y})$ in the $\sigma(A, \mathrm{sp}(P))$-topology. Furthermore, we obviously have $x_n\hat{y} \to \hat{x}\hat{y}$ in the C^*-algebra $\hat{A}$, thus applying the continuity of π again we obtain $\pi(x_n\hat{y}) \to \pi(\hat{x}\hat{y})$ in the $\sigma(A, \mathrm{sp}(P))$-topology, i.e. $\pi(\hat{x})\pi(\hat{y}) = \pi(\hat{x}\hat{y})$.

To finish the proof we refer to the well known fact that a $*$-algebra morphism from a C^*-algebra into a pre-C^*-algebra is necessarily norm continuous (cf. [1] or [2]). Thus π is norm continuous and $A = \mathrm{Ker}(\mathrm{id}_{\hat{A}} - \pi)$ is dense and closed in the C^*-algebra $\hat{A}$, i.e. $A = \hat{A}$.

III. Examples for C^*-norms defined by positive linear forms

This last section contains two important examples for C^*-norms defined by positive linear forms. Then it becomes clear that it is not difficult to construct in certain $*$-algebras with unity such sets of positive linear forms which satisfy the conditions (I)–(IV) formulated in sections I and II.

Example 1. Let A be a von Neumann algebra in the Hilbert space H and for all $z \in H$ let f_z denote the positive linear form on A defined by the formula $f_z(T) := (Tz|z)$ for all $T \in A$. Then the set $P := \{f_z | z \in H\}$ satisfies the axioms (I)–(IV).

The proof of this assertion is provided by the standard application of the Banach—Steinhaus theorem; we omit the details. In this case $\sigma(A, \mathrm{sp}(P))$ and $\sigma(A, \overline{\mathrm{sp}}(P))$ are the weak and ultraweak operator topologies on A, respectively. Of course, the norm $\| \cdot \|_P$ coincides with the usual operator norm on A.

Example 2. Let $\mathscr{A}$ denote a σ-algebra of subsets in the set T and A be the $*$-algebra of complex valued, bounded, $\mathscr{A}-\mathscr{B}(\mathbf{C})$ measurable functions (where $\mathscr{B}(\mathbf{C})$ denotes the Borel σ-algebra of $\mathbf{C}$). Let us define P as the set of integrals on A arising from σ-additive finite positive measures defined on $\mathscr{A}$. Then P satisfies the axioms (I)—(IV).

In this case $\mathrm{sp}\,(P)=\overline{\mathrm{sp}}\,(P)$ and a sequence $(\varphi_n)_{n\in\mathbf{N}}$ in A converges to the function $\varphi\in A$ in the $\sigma(A, \mathrm{sp}\,(P))$-topology if and only if $\varphi_n\to\varphi$ pointwise on T and the sequence is uniformly bounded. This proposition can be verified by the standard use of the theorem of Lebesgue and that of Banach—Steinhaus. Of course, the norm $\|\cdot\|_P$ coincides with the sup-norm.

These examples make clear that the class of C^*-algebras with unity whose norms are defined by positive linear forms satisfying the axioms (I)—(IV) contains strictly the class of von Neumann algebras. It is easy to show that there are C^*-algebras with unity (even commutative ones) that do not belong to this class. However, these C^*-algebras have certain properties very close to those of von Neumann algebras. As a matter of fact, further investigations on these C^*-algebras yield an interesting version of the spectral theorem for normal elements, analogous to the spectral theorem for normal operators in Hilbert spaces.

References

[1] J. Dixmier, *Les C*-algébres et Leurs Représentations,* Gauthier Villars Éditeurs (1968).
[2] N. Bourbaki, *Théories spectrales,* Hermann (Paris, 1967).

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